

# Casimir effect for a spherical shell in de Sitter spacetime with signature change

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## Abstract

The Casimir stress on a spherical shell in de Sitter signature changing background for massless scalar field satisfying Dirichlet boundary conditions on the shell is calculated. The Casimir stress is calculated for inside and outside of the shell with different backgrounds corresponding to different metric signatures and cosmological constants. An important contribution appears due to signature change which leads to a transient rapid expansion of the bubbles in this background.

## 1 Introduction

The Casimir effect is regarded as one of the most striking manifestation of vacuum fluctuations in quantum field theory. The presence of reflecting boundaries alters the zero-point modes of a quantized field, and results in the shifts in the vacuum expectation values of quantities quadratic in the field, such as the energy density and stresses. Therefore, the Casimir effect can be viewed as the polarization of the vacuum by boundary conditions or geometry. In particular, vacuum forces arise acting on the constraining boundaries. The particular features of these forces depend on the nature of the quantum field, the type of spacetime manifold and its dimensionality, the boundary geometries and the specific boundary conditions imposed on the field. Since the original work by Casimir in 1948 [1] many theoretical and experimental works have been done on this problem [2, 3, 4, 5, 6, 7, 8, 9].

The time dependence of boundary conditions or geometries, the so-called dynamical Casimir effect, is also a new element which has to be taken into account. In particular, in [10] the

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Casimir effect has been calculated for a massless scalar field satisfying Dirichlet boundary conditions on the spherical shell in de Sitter space. The Casimir stress is calculated for inside and outside of the shell with different backgrounds corresponding to different cosmological constants.

On the other hand, signature changing spacetimes have recently been of particular importance as specific geometries with interesting physical effects. The initial idea of signature change is due to Hartle, Hawking and Sakharov [12] which makes it possible to have a space-time with Euclidean and Lorentzian regions in quantum gravity. It has been shown that the signature change may happen even in classical general relativity [13]. The issue of propagation of quantum fields on signature-changing spacetimes has also been of some interest [14]. For example, Dray *et al* have shown that the phenomenon of particle production can happen for propagation of scalar particles in spacetime with heterotic signature. They have also obtained a rule for propagation of massless scalar fields on a two dimensional spacetime with signature change.

The Casimir effect has also recently been studied in a signature changing spacetime and shown that there is a non-vanishing pressure on the hypersurface of signature change [15]. Motivated by this new element in studying the Casimir effect we have paid attention to study such a non-trivial effect in a model of a spherical shell in de Sitter space with different cosmological constants and metric signatures inside and outside. Our aim is to understand the possible role of such a non-trivial effect in the time evolution of the bubbles at early universe with false/true vacuum and Euclidean/Lorentzian metrics, outside/inside.

## 2 Scalar Casimir effect for a sphere in de Sitter space

The Casimir force due to fluctuations of a free massless scalar field satisfying Dirichlet boundary conditions on a spherical shell in Minkowski space-time has been studied in [11]. The two-point Green's function  $G(x, t; x', t')$  is defined as the vacuum expectation value of the time-ordered product of two fields

$$G(x, t; x', t') \equiv -i < 0 | T \Phi(x, t) \Phi(x', t') | 0 > . \quad (1)$$

It has to satisfy the Dirichlet boundary conditions on the shell:

$$G(x, t; x', t')|_{|x|=a} = 0, \quad (2)$$

where  $a$  is radius of the spherical shell. The stress-energy tensor  $T^{\mu\nu}(x, t)$  is given by

$$T^{\mu\nu}(x, t) \equiv \partial^\mu \Phi(x, t) \partial^\nu \Phi(x, t) - \frac{1}{2} \eta^{\mu\nu} \partial_\lambda \Phi(x, t) \partial^\lambda \Phi(x, t). \quad (3)$$

The radial Casimir force per unit area  $\frac{F}{A}$  on the sphere, called Casimir stress, is obtained from the radial-radial component of the vacuum expectation value of the stress-energy tensor:

$$\frac{F}{A} = \langle 0 | T_{in}^{rr} - T_{out}^{rr} | 0 \rangle |_{r=a}. \quad (4)$$

Taking into account the relation (1) between the vacuum expectation value of the stress-energy tensor  $T^{\mu\nu}(x, t)$  and the Green's function at equal times  $G(x, t; x', t)$  we obtain

$$\frac{F}{A} = \frac{i}{2} \left[ \frac{\partial}{\partial r} \frac{\partial}{\partial r'} G(x, t; x', t)_{in} - \frac{\partial}{\partial r} \frac{\partial}{\partial r'} G(x, t; x', t)_{out} \right]_{x=x', |x|=a}. \quad (5)$$

One may use of the above flat space calculation in de Sitter space-time by taking the de Sitter metric in conformally flat form

$$ds^2 = \Omega(\eta) \left[ d\eta^2 - \sum_{i=1}^3 (dx^i)^2 \right], \quad (6)$$

where  $\Omega(\eta) = \frac{a}{\eta}$  and  $\eta$  is the conformal time

$$-\infty < \eta < 0. \quad (7)$$

Assuming a canonical quantization of the scalar field, the conformally transformed quantized scalar field in de Sitter space is given by

$$\bar{\Phi}(x, \eta) = \sum_k [a_k \bar{u}_k(\eta, x) + a_k^\dagger \bar{u}_k^*(\eta, x)], \quad (8)$$

where  $a_k^\dagger$  and  $a_k$  are creation and annihilation operators respectively and the vacuum states associated with the modes  $\bar{u}_k$  defined by  $a_k |\bar{0}\rangle = 0$ , are called conformal vacuum. Given the flat space Green's function(1), we obtain

$$\bar{G} = -i \langle \bar{0} | T \bar{\Phi}(x, \eta) \bar{\Phi}(x', \eta') | \bar{0} \rangle = \Omega^{-1}(\eta) \Omega^{-1}(\eta') G, \quad (9)$$

where  $\bar{\Phi}(x, \eta) = \Omega^{-1}(\eta) \Phi(x, \eta)$  has been used. Therefore, using Eqs.(4), (5) and (9) we obtain the total stress on the sphere in de Sitter space as

$$\left( \frac{\bar{F}}{A} \right) = \frac{\eta^2}{\alpha^2} \frac{F}{A}. \quad (10)$$

### 3 Spherical shell with different vacua and signatures inside and outside

We assume different vacua inside and outside, corresponding to different  $\alpha_{in}$  and  $\alpha_{out}$  for the Lorentzian metric (6) and use the following relation for the stress on the shell due to boundary conditions in flat spacetime [8]

$$\frac{F}{A} = \frac{-1}{4\pi a^2} \frac{\partial E}{\partial a}, \quad (11)$$

where the Casimir energy  $E$  is the sum of Casimir energies  $E_{in}$  and  $E_{out}$  for inside and outside of the shell. The corresponding relation in de Sitter space is given by [10]

$$\frac{\bar{F}}{A} = \frac{-1}{4\pi a^2} \frac{\partial \bar{E}}{\partial a} = \frac{\eta^2}{8\pi a^4} \left( \frac{c_1}{\alpha_{in}^2} + \frac{c_2}{\alpha_{out}^2} \right). \quad (12)$$

where we have used the renormalized total zero-point energy

$$\bar{E} = \frac{\eta^2}{2a} \left( \frac{c_1}{\alpha_{in}^2} + \frac{c_2}{\alpha_{out}^2} \right). \quad (13)$$

in which  $c_1 = 0.008873$ ,  $c_2 = -0.003234$ .

Now, we obtain the pure effect of vacuum polarization due to the gravitational field without any boundary conditions in Euclidean (outside) region with the following metric

$$ds^2 = -\Omega(\eta)[d\eta^2 + \sum_{i=1}^3 (dx^i)^2]. \quad (14)$$

To this end, we calculate the renormalized stress tensor for the massless scalar field in de Sitter space with Euclidean signature. One may use [2]

$$\langle 0|T_\mu^\nu[g_{kl}]|0\rangle|_{ren} = (\tilde{g}/g)^{1/2} \langle 0|T_\mu^\nu[\tilde{g}_{kl}]|0\rangle|_{ren} - \frac{1}{2880\Pi^2} \left[ \frac{1}{6} {}^{(1)}H_\mu^\nu - {}^{(3)}H_\mu^\nu \right], \quad (15)$$

where  $\tilde{g}_{kl}$  is the flat Euclidean metric for which  $\langle 0|T_\mu^\nu[\tilde{g}_{kl}]|0\rangle|_{ren} = 0$ , and

$$\begin{aligned} {}^{(1)}H_\mu^\nu &= 0, \\ {}^{(3)}H_\mu^\nu &= \frac{3}{\alpha^4} \delta_\mu^\nu. \end{aligned}$$

We then obtain

$$\langle 0|T_\mu^\nu[g_{kl}]|0\rangle|_{ren} = \frac{1}{960\Pi^2\alpha^4} \delta_\mu^\nu, \quad (16)$$

which is exactly the same result for the Lorentzian case [2]. Therefore, the corresponding effective radial pressures for the Euclidean (outside) and Lorentzian (inside) regions with  $\alpha_{out}$  and  $\alpha_{in}$ , due to pure effect of gravitational vacuum polarization without any boundary condition, are given respectively by

$$\begin{aligned} P_{out}^E &= -\langle 0|T_r^r[g_{kl}]|0\rangle|_{ren} = -\frac{1}{960\Pi^2\alpha_{out}^4} \\ P_{in}^L &= -\langle 0|T_r^r[g_{kl}]|0\rangle|_{ren} = -\frac{1}{960\Pi^2\alpha_{in}^4}. \end{aligned}$$

The corresponding gravitational pressure on the spherical shell is then given by

$$P_G = P_{in}^L - P_{out}^E = -\frac{1}{960\pi^2} \left( \frac{1}{\alpha_{in}^4} - \frac{1}{\alpha_{out}^4} \right). \quad (17)$$

We now proceed to calculate the stress due to the boundary effects  $P_B$ . To this end, we make maximum use of the results obtained in [10]. The stress on the shell due to boundary effects for the Lorentzian metric (6) has been obtained as (12). In signature changing case we have correspondingly

$$\left( \frac{\tilde{F}}{A} \right)_{L-E} = \langle 0|T_{rr}|0\rangle|_{in}^L - \langle 0|T_{rr}|0\rangle|_{out}^E, \quad (18)$$

where

$$\langle 0|T_{\mu\nu}|0\rangle = \sum_{\alpha} T_{\mu\nu}\{\Phi_{\alpha}, \Phi_{\alpha}^*\}. \quad (19)$$

The detailed calculations show that the only difference between the Lorentzian case and signature changing one appears as follows

$$\left(\frac{\tilde{F}}{A}\right)_{L-E} = \left(\frac{\tilde{F}}{A}\right)_L + \partial_{\eta}\partial^{\eta}\{\Phi_{\alpha_{in}}, \Phi_{\alpha_{in}}^*\}_L - \partial_{\eta}\partial^{\eta}\{\Phi_{\alpha_{out}}, \Phi_{\alpha_{out}}^*\}_E. \quad (20)$$

The scalar field  $\Phi(r, \theta, \eta)$  in the Lorentzian de Sitter space satisfies

$$(\square + \xi R)\Phi(r, \theta, \eta) = 0, \quad (21)$$

where  $\square$  is the Laplace-Beltrami operator for the de Sitter metric, and  $\xi$  is the coupling constant. For conformally coupled field in four dimension  $\xi = \frac{1}{6}$ , and  $R$ , the Ricci scalar curvature, is given by

$$R = 12\alpha^{-2}. \quad (22)$$

Taking into account the separation of variables as

$$\Phi_L(r, \theta, \eta) = A(r)B(\theta)T_L(\eta), \quad (23)$$

for the inside Lorentzian domain with

$$T_L(\eta) = \exp^{-i\omega\eta}, \quad (24)$$

the corresponding Euclidean  $\eta$ -dependence takes on the form

$$T_E(\eta) = \exp^{-\omega\eta}, \quad (25)$$

for the scalar field to be normalizable in  $\eta$ . Inserting Eqs.(24) and (25) into Eq.(20) for  $\Phi_L(r, \theta, \eta)$  and  $\Phi_E(r, \theta, \eta)$ , respectively we obtain

$$\left(\frac{\tilde{F}}{A}\right)_{L-E} = \frac{\eta^2}{8\pi a^4} \left(\frac{c_1}{\alpha_{in}^2} + \frac{c_2}{\alpha_{out}^2}\right) + \frac{\alpha_{out}^2}{\eta^2} (8\omega^2 e^{-2\omega\eta}). \quad (26)$$

Taking into account the gravitational pressure on the shell we obtain the total result

$$P = P_G + P_B = -\frac{1}{960\pi^2} \left(\frac{1}{\alpha_{in}^4} - \frac{1}{\alpha_{out}^4}\right) + \frac{\eta^2}{8\pi a^4} \left(\frac{c_1}{\alpha_{in}^2} + \frac{c_2}{\alpha_{out}^2}\right) + \frac{\alpha_{out}^2}{\eta^2} (8\omega^2 e^{-2\omega\eta}). \quad (27)$$

Inserting  $\alpha^2 = \frac{3}{\Lambda}$  we obtain the total pressure in terms of the cosmological constants

$$P = -\frac{1}{2880\pi^2} (\Lambda_{in}^2 - \Lambda_{out}^2) + \frac{\eta^2}{24\pi a^4} (c_1 \Lambda_{in} + c_2 \Lambda_{out}) + \frac{3}{\Lambda_{out} \eta^2} (8\omega^2 e^{-2\omega\eta}). \quad (28)$$

The first two terms are not new ( see [10]), but the third one shows a nontrivial effect due to signature changing background. The first term  $P_G$  which is pure gravitational effect without boundary is  $\eta$ -independent, but is sensitive to the initial values of  $\Lambda_{in}$  and  $\Lambda_{out}$ . In the

inflationary interesting case, namely the true vacuum inside and false vacuum outside, i.e.  $\Lambda_{in} < \Lambda_{out}$ , this pressure is always repulsive. However, the last two terms are  $\eta$ -dependent resulting from boundary effects. So, one has to proceed cautiously. Let us assume

$$c_1\Lambda_{in} + c_2\Lambda_{out} > 0. \quad (29)$$

This leads the second term to be an ever increasing repulsive pressure with time  $\eta$ , so that the pressure due to first two terms is always positive. On the other hand, taking

$$c_1\Lambda_{in} + c_2\Lambda_{out} < 0, \quad (30)$$

with  $|c_1| > |c_2| = -c_2$ , the pressure due to first two terms may be either negative or positive. If this pressure is initially positive, the initial repulsion of the expanding bubble will be stopped at an specific time  $\eta$  and then a negative pressure leads to a contracting shell ending up with a collapse of the bubble. If on the other hand, this pressure is initially negative it remains negative forever. Now, we consider the third term which is of particular importance in the present work. This term acts as dynamically decaying pressure which is very large at the beginning  $\eta \simeq 0$ , but is vanishing at late times  $\eta \gg 0$ . Moreover, it is positive provided the false vacuum  $\Lambda_{out} > 0$ , which is reasonable in the context of inflationary models. The existence of this new dynamical pressure may play an important role in the inflationary phase at early universe.

In case of (30), the general behavior of the bubble is not affected by the third term because it is transient and its presence merely alters the time  $\eta$  of the contracting phase and the final collapse of the bubble.

However, in case of (29) which is of particular interest in inflationary phase at early universe, all contributions are positive and the bubble initially ( $\eta \simeq 0$ ) experiences a huge repulsion mainly due to the third term. At late times  $\eta \gg 0$  the third term almost vanishes, and the second term plays the important role in the late time repulsion of the bubble.

## 4 Conclusion

We have studied the Casimir effect for spherical bubbles with different vacua and metric signatures inside and outside, corresponding to de Sitter metrics. The metrics inside and outside are taken Lorentzian and Euclidean, respectively. The case of different vacua in a Lorentzian de Sitter spacetime has already been studied in [10]. In the present work we have shown how those results are affected in a signature changing background. It is shown that a transient term appears due to signature change, which in comparison to the Lorentzian case ( true vacuum inside and false vacuum outside ) results initially in an extra rapid expansion of the bubble. This extra rapid expansion vanishes at late times and the Lorentzian results are then recovered. It reveals that the presence of signature change may cause to more rapid transient expansion of bubbles than is predicted by Lorentzian case in [10]. This may have some important impacts on the inflationary phase at early universe.

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